Continuous Functions

Definition (Continuous functions) A function f is continuous at z_0 if

$$\lim_{z \to z_0} f(z) = f(z_0).$$

That is: for all $\varepsilon > 0$, $\exists \delta > 0$ such that

$$|z - z_0| < \delta \implies |f(z) - f(z_0)| < \varepsilon.$$

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By the limit laws, sums, products, and quotients of continuous functions are continuous (where they are defined).

Theorem (Composition of continuous functions) Suppose that f is defined on an open disk $D_\varepsilon(z_0)$ and the domain of g contains $f(D_\varepsilon(z_0))$. If f is continuous at z_0 and g is continuous at $f(z_0)$, then $g \circ f$ is continuous at z_0.

Proof. Let $\varepsilon > 0$. Choose $\delta_1 > 0$ such that

$$|w - f(z_0)| < \delta_1 \implies |g(w) - g(f(z_0))| < \varepsilon.$$

Choose $\delta_2 > 0$ such that

$$|z - z_0| < \delta_2 \implies |f(z) - f(z_0)| < \delta_1$$

Now, $|z - z_0| < \delta_2$ implies $|g(f(z)) - g(f(z_0))| < \varepsilon$.

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Theorem If f is continuous and nonzero at z_0, then there exists $\varepsilon > 0$ such that $f(z) \neq 0$ for all $z \in D_\varepsilon(z_0)$.

Proof. Suppose f is continuous and nonzero at z_0. Then $|f(z_0)| > 0$. Take $\varepsilon = \frac{|f(z_0)|}{2}$. Suppose $f(z) = 0$ for some $z \in D_\varepsilon(z_0)$. Then by continuity at z_0,

$$0 < |f(z_0)| = |f(z) - f(z_0)| < \varepsilon = \frac{|f(z_0)|}{2}.$$

Contradiction!

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$\boxed{\text{Theorem (Continuity in Terms of Re } f/\text{Im } f)}$ Suppose that

$$f(z) = u(x,y) + i\,v(x,y).$$

Then f is continuous at $z_0 = x_0 + iy_0$ if and only if both u and v are continuous at (x_0, y_0).

Proof. Follows from theorem on limits in terms of Re f/Im f.

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A subset of $\mathbb{C}$ is compact if it is closed and bounded. A function $f : \Omega \xrightarrow{\subseteq \mathbb{C}} \mathbb{C}$ is bounded if there exists $M \geq 0$ such that $|f(z)| \leq M$ for all $z \in \Omega$.

$\boxed{\text{Theorem (Extreme Value theorem)}}$ If R is a compact set and $f : R \to \mathbb{C}$ is continuous on R, then f is bounded and it achieves this bound.

Proof. If $f = u(x,y) + i\,v(x,y)$ is continuous, then $u, v : R \xrightarrow{\subseteq \mathbb{R}^2} \mathbb{R}$ are continuous on R. Hence, so is

$$|f(z)| = \sqrt{u(x,y)^2 + v(x,y)^2}.$$

By vector calc, $|f(z)|$ is bound and achives its bound.

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Differentiable Functions

$\boxed{\text{Definition (Derivative)}}$ Suppose the domain of definition of f contains an open disk $D_\varepsilon(z_0)$. The derivative of f at z_0 is the limit

$$\frac{df}{dz}(z_0) = f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

When the limit exists, f is differentiable. Letting $\Delta z = z - z_0$, this can also be written

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}.$$

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$\boxed{\text{Example}}$ $f(z) = z^2$. We have

$$f'(z) = \lim_{\Delta z \to 0} \frac{(z+\Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z \to 0} \frac{2z\Delta z + \Delta z^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} 2z + \Delta z$$

$$= 2z.$$

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Sometimes it will be convenient to use the notation

$$\Delta w = f(z + \Delta z) - f(z)$$

So that

$$f'(z) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}.$$

$\boxed{\text{Example}}$ Where is $f(z) = |z|^2$ differentiable? Let $z \in \mathbb{C}$.

Compute
$$\begin{aligned}
\Delta w &= |z + \Delta z|^2 - |z|^2 \\
&= (z + \Delta z)\overline{(z + \Delta z)} - |z|^2 \\
&= \cancel{z\bar{z}} + z\overline{\Delta z} + \Delta z \bar{z} + \Delta z \overline{\Delta z} - \cancel{|z|^2} \\
&= z\overline{\Delta z} + \Delta z \bar{z} + \Delta z \overline{\Delta z}.
\end{aligned}$$

Then
$$\frac{\Delta w}{\Delta z} = \frac{z\overline{\Delta z}}{\Delta z} + \bar{z} + \overline{\Delta z}.$$

Along the real axis $\Delta z = \overline{\Delta z}$. So

$$\frac{\Delta w}{\Delta z} = z + \bar{z} + \overline{\Delta z}$$

as $\Delta z \to 0$ the limit is $z + \bar{z}$. Along the imaginary axis, $\Delta z = -\overline{\Delta z}$ so

$$\frac{\Delta w}{\Delta z} = \bar{z} - z - \overline{\Delta z}.$$

As $\Delta z \to 0$ the limit is $\bar{z} - z$. Since limits are unique, if $f'(z)$ exists, then $\bar{z} - z = z + \bar{z}$. Hence $z = 0$. Does $f'(0)$ exist? When $z = 0$

$$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \overline{\Delta z} = 0.$$

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The preceding example shows two surprising facts:

(1) f' can exist at a single point and nowhere else in a neighborhood of that point.

(2) $\mathrm{Re}\, f / \mathrm{Im}\, f$ can have continuous partial derivatives of all orders, and yet f' does not exist.
Note: $\mathrm{Re}(|f|^2) = x^2 + y^2$ and $\mathrm{Im}(|f|^2) = 0$.

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$\boxed{\text{Proposition (Differentiable functions are Continuous)}}$ If f is differentiable at z_0, then f is continuous at z_0.

Proof. Suppose f is differentiable at z_0. Then

$$\lim_{z \to z_0} \left(f(z) - f(z_0) \right) = \left(\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \right) \underbrace{\left(\lim_{z \to z_0} z - z_0 \right)}_{=0} = 0.$$

Hence, $\lim_{z \to z_0} f(z) = f(z_0)$.

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$\boxed{\text{Proposition (Differentiation Laws)}}$ Suppose f and g are differentiable at z. Then

(1) $\frac{d}{dz} c = 0, \ \forall c \in \mathbb{C}$

(2) $\frac{d}{dz}(c f(z)) = c f'(z), \ \forall c \in \mathbb{C}$ (Constant Rule)

(3) $\frac{d}{dz} z^n = n z^{n-1}, \ \forall n \in \mathbb{N}$ (Power Rule)

(4) $\frac{d}{dz}\left(f(z)+g(z)\right) = f'(z) + g'(z)$ (Sum Rule)

(5) $\frac{d}{dz}\left(f(z)g(z)\right) = f'(z)g(z) + g'(z)f(z)$ (Product Rule)

(6) $\frac{d}{dz}\left(\frac{f(z)}{g(z)}\right) = \frac{f'(z)g(z) - g'(z)f(z)}{g(z)^2}$, $g(z) \neq 0$ (Quotient Rule)

Proof.

(5) Compute

$$\begin{aligned}\Delta w &= f(z+\Delta z)g(z+\Delta z) - f(z)g(z)\\ &= f(z+\Delta z)g(z+\Delta z) - f(z+\Delta z)g(z) + f(z+\Delta z)g(z) - f(z)g(z)\\ &= f(z+\Delta z)\left(g(z+\Delta z)-g(z)\right) + g(z)\left(f(z+\Delta z)-f(z)\right).\end{aligned}$$

So
$$\begin{aligned}\lim_{\Delta z\to 0}\frac{\Delta w}{\Delta z} &= \underbrace{\lim_{\Delta z\to 0} f(z+\Delta z)}_{= f(z) \text{ since } f \text{ is continuous}}\ \lim_{\Delta z\to 0}\frac{g(z+\Delta z)-g(z)}{\Delta z} + \lim_{\Delta z\to 0} g(z) \lim_{\Delta z\to 0}\frac{f(z+\Delta z) - f(z)}{\Delta z}\\ &= f(z)g'(z) + g(z)f'(z).\end{aligned}$$

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$\boxed{\text{Proposition (Chain Rule)}}$ Suppose that f is differentiable at z_0 and g is differentiable at $f(z_0)$. Then $g\circ f$ is differentiable at z_0 and

$$(g\circ f)'(z_0) = g'(f(z_0))\, f'(z_0).$$

Proof. Since $g'(f(z_0))$ exists, there is open disk $D_\varepsilon(f(z_0))$ on which g is defined. On this disk define a function

$$\Phi(w) = \begin{cases} (*)\ \dfrac{g(w)-g(f(z_0))}{w-f(z_0)} - g'(f(z_0)), & \text{if } w \neq f(z_0)\\ 0, & \text{if } w = f(z_0)\end{cases}$$

Notice $\lim_{w\to f(z_0)} \Phi(w) = 0 = \Phi(f(z_0))$. Hence, $\Phi(w)$ is continuous at $f(z_0)$. Rewrite $(*)$ as

$$(*)\quad g(w) - g(f(z_0)) = \Big(\Phi(w) + g'(f(z_0))\Big)\Big(w - f(z_0)\Big).$$

Now, since f is continuous at z_0 choose $\delta > 0$ such that

$|z - z_0| < \delta$ implies $|f(z) - f(z_0)| < \varepsilon$, that is $f(z) \in D_\varepsilon(f(z_0))$.

Now, when $|z - z_0| < \delta$, we can take $w = f(z)$ in $(*)$ and divide by $z - z_0$:

$$\frac{g(f(z)) - g(f(z_0))}{z - z_0} = \Big(\Phi(f(z)) + g'(f(z_0))\Big)\Big(\frac{f(z) - f(z_0)}{z - z_0}\Big).$$

Taking the limit $z \to z_0$ we get

$$g'(f(z)) = \big(0 + g'(f(z_0))\big) f'(z_0)$$
$$= g'(f(z_0))\, f'(z_0).$$

Φ's continuous

Since $\lim_{z\to z_0} \Phi(f(z)) = \Phi\left(\lim_{z\to z_0} f(z)\right)$

f is continuous $\rightarrow$ $= \Phi(f(z_0)) = 0$ — definition of Φ

$\blacksquare$

Cauchy-Riemann Equations

Writing $z = x + iy$ and $\Delta z = \Delta x + i\Delta y$, we compute:

$$\frac{\Delta w}{\Delta z} = \frac{u(x+\Delta x, y+\Delta y) + i v(x+\Delta x, y+\Delta y) - (u(x,y) + i v(x,y))}{\Delta x + i\Delta y}$$
$$= \frac{u(x+\Delta x, y+\Delta y) - u(x,y)}{\Delta x + i\Delta y} + i\left(\frac{v(x+\Delta x, y+\Delta y) - v(x,y)}{\Delta x + i\Delta y}\right).$$

Along the real axis, $\Delta y = 0$ so we get

$$f'(z) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i \lim_{\Delta x\to 0} \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}$$
$$= u_x(x,y) + i\, v_x(x,y).$$

Along the imaginary axis, $\Delta x = 0$ so we get

$$f'(z) = \lim_{\Delta y \to 0} \frac{u(x,y+\Delta y) - u(x,y)}{i \Delta y} + i \lim_{\Delta y \to 0} \frac{v(x,y+\Delta y) - v(x,y)}{i \Delta y}$$

$$= \frac{1}{i} u_y + \frac{i}{i} v_y = v_y(x,y) - i u_y(x,y) \quad \left(\frac{1}{i} = -i\right)$$

So, since limits are unique we get

$$u_x + i v_x = v_y - i u_y$$

Hence, compare Re/Im part:

$$\begin{cases} u_x(x,y) = v_y(x,y) \\ u_y(x,y) = -v_x(x,y) \end{cases}$$

Cauchy - Riemann Equations

$\square$

We just proved:

Theorem (Cauchy - Riemann Equations) Suppose that

$$f(z) = u(x,y) + i v(x,y)$$

is differentiable at $z = x + iy$. Then

(1) the first order partial derivatives of u and v exist and satisfy the Cauchy - Riemann Equations

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$

(2) $f'(z) = u_x(x,y) + i v_x(x,y) = v_y(x,y) - i v_x(x,y)$. //

The CR-equations are a necessary condition for f' to exist. We can use them to locate some points where the derivative does not exist.

Example $f(z) = |z|^2 = x^2 + y^2 + i\,0$. Note $u(x,y) = x^2 + y^2$ and $v(x,y) = 0$,

We have $u_x(x,y) = 2x \qquad v_x(x,0) = 0$

$u_y(x,y) = 2y \qquad v_y(0,0) = 0$

The CR-Riemann equations:

$$2x = u_x = v_y = 0$$

$$2y = u_y = -v_x = 0,$$

So $x=0$ and $y=0$.

So $f'(z)$ does not exist when $z \neq 0$. //

Note: this doesn't show that $f'(0)$ exists.

The Cauchy-Riemann Equations are not sufficient for the existence of the derivative, as the next example shows.

$\boxed{\text{Example}}$ Suppose $f(z) = \begin{cases} \frac{\bar{z}^2}{z}, & z \neq 0 \\ 0, & z = 0 \end{cases}$. Then

$$u(x,y) = \begin{cases} \frac{x^3 - 3xy^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases} \quad \text{and} \quad v(x,y) = \begin{cases} \frac{y^3 - 3x^2 y}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

We show that u, v satisfy the CR-eq at 0.

$$u_x(0,0) = \lim_{\Delta x \to 0} \frac{u(0+\Delta x, 0) - u(0,0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\frac{\Delta x^3}{\Delta x^2} - 0}{\Delta x} = 1$$

$$u_y(0,0) = \lim_{\Delta y \to 0} \frac{u(0, 0+\Delta y) - u(0,0)}{\Delta y} = 0.$$

$$v_x(0,0) = \lim_{\Delta x \to 0} \frac{v(0+\Delta x, 0) - v(0,0)}{\Delta x} = 0$$

$$v_y(0,0) = \lim_{\Delta y \to 0} \frac{v(0, 0+\Delta y) - v(0,0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{\frac{\Delta y^3}{\Delta y^2} - 0}{\Delta y} = 1.$$

Hence
$$u_x(0,0) = 1 = v_y(0,0)$$
$$u_y(0,0) = 0 = -0 = -v_x(0,0)$$

So CR-eq are satisfied. But $f'(0)$ does not exist (exercise). //

$\boxed{\text{Theorem (Sufficient Condition for Differentiability)}}$ Suppose that $f(z) = u(x,y) + i v(x,y)$ is defined on a neighborhood of $z = x+iy$. If (1) the first order partial derivatives of u, v exist everywhere in the neighborhood;

(2) the partial derivatives are continuous and satisfy the CR-equations at (x,y);

then $f'(z)$ exists and is given by $f'(z) = u_x(x,y) + i\, v_x(x,y)$.

Proof. See the book. $\square$

Example

$x \in \mathbb{R}$, so e^x is the usual exponential

$iy \notin \mathbb{R}$ so e^{iy} is defined by Euler's formula

(1) $f(z) = e^x e^{iy} = e^x \cos y + i\, e^x \sin y$. Note that $u(x,y) = e^x \cos y$, $v(x,y) = e^x \sin y$ have continuous partial derivatives on all of $\mathbb{R}^2$. Moreover

$$u_x = e^x \cos y = v_y$$
$$u_y = -e^x \sin y = -v_x.$$

So CR-eq are satisfied everywhere $\Rightarrow$ $f'(z)$ exists everywhere on $\mathbb{C}$. //